

List coloring of matroids and base exchange properties

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ABSTRACT. A coloring of a matroid is an assignment of colors to the elements of its ground set. We restrict to proper colorings – those for which elements of the same color form an independent set. Seymour proved that a k -colorable matroid is also colorable from any lists of size k .

We generalize this theorem to the case when lists have still fixed sizes, but not necessarily equal. For any fixed size of lists assignment ℓ , we prove that, if a matroid is colorable from a particular lists of size ℓ , then it is colorable from any lists of size ℓ . This gives an explicit necessary and sufficient condition for a matroid to be list colorable from any lists of a fixed size.

As an application, we show how to use our condition to derive several base exchange properties.

1. Introduction

Let M be a matroid on a ground set E (we refer the reader to [5] for a background of matroid theory). A *coloring* of M is an assignment of colors to the elements of E . In analogy to graph theory we say that a coloring is *proper* if elements of the same color form an independent set in the matroid. Via this correspondence one can define for matroids all chromatic parameters studied for graphs.

The *chromatic number* of a loopless matroid M , denoted by $\chi(M)$, is the minimum number of colors in a proper coloring of M . For instance, if M is a graphic matroid obtained from a graph G , then $\chi(M)$ is the least number of colors needed to color edges of G so that no cycle is monochromatic. This number is known as the *arboricity* of the underlying graph G .

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For matroids the chromatic number can be easily expressed in terms of rank function. Extending a theorem of Nash-Williams [8] for graph arboricity, Edmonds [2] proved the following formula

$$\chi(M) = \max_{\emptyset \neq A \subseteq E} \left\lceil \frac{|A|}{r(A)} \right\rceil,$$

where r is the rank function of a loopless matroid M on a ground set E . Some game-theoretic versions of the chromatic number of a graph were already studied for matroids, see [6, 7].

In this note we study list coloring of matroids. The concept of list coloring was initiated for graphs by Vizing [10], and independently by Erdős, Rubin and Taylor [3]. Let us recall its definition in the matroid setting.

Suppose each element e of the ground set E of a matroid M is assigned with a set (a *list*) of colors $L(e)$. By the *size of list assignment* (or simply *lists*) L we mean the function ℓ satisfying $\ell(e) = |L(e)|$ for each $e \in E$. We say that matroid M is *colorable from lists* L if there exists a proper coloring of M , such that each element receives a color from its list. The *list chromatic number* (sometimes called *choice number*) of a loopless matroid M , denoted by $\text{ch}(M)$, is the minimum number k , such that M is colorable from any lists of size at least k .

Clearly, $\text{ch}(M) \geq \chi(M)$. For graphs in general, the inequality between corresponding parameters is strict ($\text{ch}(G)$ is not even bounded by a function of $\chi(G)$). A celebrated result of Galvin [4] asserts that for the line graph of a bipartite graph there is an equality. Surprisingly, Seymour [9] proved that actually equality holds for all matroids.

THEOREM 1. *For every loopless matroid M there is an equality $\text{ch}(M) = \chi(M)$.*

Seymour's theorem can be rephrased by saying that the following conditions are equivalent:

- (1) matroid M is colorable from the lists $L(e) = \{1, \dots, k\}$,
- (2) matroid M is colorable from any lists of size k .

Our main result is a generalization of Seymour's Theorem 1 to the setting, where sizes of lists are still fixed, but not necessarily equal.

THEOREM 2. *Let M be a matroid and let ℓ be a lists size function. Then the following conditions are equivalent:*

- (1) *matroid M is colorable from the lists $L_\ell(e) = \{1, \dots, \ell(e)\}$,*
- (2) *matroid M is colorable from any lists of size ℓ .*

As a corollary we get a strengthening of Seymour's Theorem 1. Namely, a k -colorable matroid is also colorable from any lists of fixed size varying between 1 and k , and average size at most $\frac{k+1}{2}$.

COROLLARY 3. *Let M be a k -colorable matroid, and let I_1, \dots, I_k be a partition of its ground set into independent sets (color classes). Define lists size function by $\ell(e) = i$ for elements $e \in I_i$. Then matroid M is colorable from any lists of size ℓ .*

In the last section we make a link between list coloring and base exchange properties. We use our theorem as a tool to obtain easily several such properties. The idea is to choose suitable lists, such that existence of a proper coloring guarantees a particular exchange property. The crucial point is that lists may have different sizes (ex. if size of a list is 1, then a color is already determined).

2. Proof of Theorem 2

PROOF. Clearly, condition (1) follows from (2). We argue the opposite implication. Let L be a fixed list assignment of size ℓ . Without loss of generality we can assume that all lists $L(e)$ are subsets of a finite set of integers $\{1, \dots, d\}$. Let us denote $Q_i = \{e \in E : i \in L(e)\}$, and respectively Q_i^ℓ for lists L_ℓ .

Consider matroids M_1, \dots, M_d , with M_i equal to the restriction $M|_{Q_i}$ of M to the set Q_i (with the ground set trivially extended to E). It is straightforward that a proper coloring from lists L exists if and only if it is possible to partition the ground set E into subsets I_1, \dots, I_d with I_i independent in the matroid M_i (I_i is the color class of i). By the Matroid Union Theorem (see [5]) such a partition exists if and only if for every subset $A \subset E$ there is an inequality

$$r(A \cap Q_1) + \dots + r(A \cap Q_d) \geq |A|.$$

Analogously, a proper coloring from lists L_ℓ exists if and only if for every subset $A \subset E$ there is an inequality

$$r(A \cap Q_1^\ell) + \dots + r(A \cap Q_d^\ell) \geq |A|.$$

Thus, to prove that condition (1) implies (2) it is enough to show that for every subset $A \subset E$ there is an inequality

$$(2.1) \quad r(A \cap Q_1) + \dots + r(A \cap Q_d) \geq r(A \cap Q_1^\ell) + \dots + r(A \cap Q_d^\ell).$$

Notice that $\bigcup_i Q_i$ and $\bigcup_i Q_i^\ell$ are equal as multisets, since both L and L_ℓ are list assignments of size ℓ (each $e \in E$ belongs to each of the unions exactly $\ell(e)$ times). We will show that the inequality (2.1) is satisfied for any sets Q_i satisfying $\bigcup_i Q_i = \bigcup_i Q_i^\ell$ as multisets. The proof is by induction on the number of pairs of sets Q_k, Q_l such that Q_k and Q_l are incomparable in the inclusion order (none is contained in the other).

If the number of such pairs is zero, then Q_i are linearly ordered by inclusion. Let us reorder them in such a way that $Q_1 \supset Q_2 \supset \dots \supset Q_d$. Then the equality $\bigcup_i Q_i = \bigcup_i Q_i^\ell$ implies that $Q_1 = Q_1^\ell, Q_2 = Q_2^\ell, \dots, Q_d = Q_d^\ell$, so the inequality (2.1) is in fact an equality.

Suppose now that there exists a pair of sets Q_k, Q_l incomparable in the inclusion order. Replace them in the family $\{Q_i\}_{i=1, \dots, d}$ by sets $Q_k \cup Q_l$ and $Q_k \cap Q_l$ to obtain a family $\{Q'_i\}_{i=1, \dots, d}$. Since $Q_k \cup Q_l = (Q_k \cup Q_l) \cup (Q_k \cap Q_l)$ as multisets, the sets Q'_i also satisfy the multiset equality $\bigcup_i Q'_i = \bigcup_i Q_i^\ell$. Moreover, the number of pairs incomparable in the inclusion order among Q'_i is lower than among sets Q_i . By the inductive assumption, the inequality (2.1) holds for sets Q'_i . Combining it with submodularity of the rank function

$$r(A \cap Q_k) + r(A \cap Q_l) \geq r(A \cap (Q_k \cup Q_l)) + r(A \cap (Q_k \cap Q_l)),$$

we get inequality (2.1) for sets Q_i . This completes the inductive step. \square

3. Applications

A family \mathcal{B} of subsets of a finite set E , just from the definition, forms a set of bases of a matroid if it is non-empty, and if for every $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that $B_1 \cup f \setminus e \in \mathcal{B}$.

In this case a stronger property holds. For every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both $B_1 \cup f \setminus e$ and $B_2 \cup e \setminus f$ are bases. It is called *symmetric exchange property*, and was discovered by Brualdi [1].

Surprisingly, even more is true. One can exchange symmetrically not only single elements, but also subsets. This property is known as *multiple symmetric exchange*. We demonstrate usefulness of Theorem 2 by using it as a tool to give easy proofs of multiple symmetric exchange property and its generalizations.

PROPOSITION 4. *Let B_1 and B_2 be bases of a matroid M . Then for every $A_1 \subset B_1$ there exists $A_2 \subset B_2$, such that $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are both bases.*

PROOF. Observe that by adding parallel elements to the elements of the intersection $B_1 \cap B_2$ we can restrict to the case when bases B_1 and B_2 are disjoint.

When bases B_1, B_2 are disjoint, then restrict matroid M to their union. Let ℓ be a lists size function such that $\ell|_{B_1} \equiv 1$, and $\ell|_{B_2} \equiv 2$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let L be a list assignment of size ℓ which assigns list $\{1\}$ to elements of A_1 , list $\{2\}$ to elements of $B_1 \setminus A_1$, and list $\{1, 2\}$ to elements of B_2 . By Theorem 2 there exists a proper coloring from these lists. Denote by C_1 elements colored with 1, and by C_2 those colored with 2. Now $A_2 = C_2 \cap B_2$ is a good choice, since sets $(B_1 \setminus A_1) \cup A_2 = C_2$ and $(B_2 \setminus A_2) \cup A_1 = C_1$ are independent. \square

Multiple symmetric exchange property can be slightly generalized. Instead of having a partition of one of bases into two parts we can have an arbitrary partition of it. We prove that for any such partition there exists a partition of the second basis which is consistent in two different ways.

PROPOSITION 5. *Let A and B be bases of a matroid M . Then for every partition $B_1 \sqcup \dots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \dots \sqcup A_k = A$, such that $(B \setminus B_i) \cup A_i$ are bases for all $1 \leq i \leq k$.*

PROOF. As before we can assume that bases A, B are disjoint, and restrict to their union. Consider matroid M' equal to M , where elements of B have $k - 1$ parallel copies. So the ground set of M' equals to $B^1 \sqcup \dots \sqcup B^{k-1} \sqcup A$. Let ℓ be a lists size function such that $\ell|_{B^i} \equiv k - 1$, and $\ell|_A \equiv k$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let L be a list assignment of size ℓ which assigns list $\{1, \dots, k\} \setminus \{i\}$ to elements of B_i and all its copies, and list $\{1, \dots, k\}$ to elements of A . By Theorem 2 there exists a proper coloring from these lists. Denote by C_i elements colored with i . Now $A_i = C_i \cap A$ is a good partition, since sets $(B \setminus B_i) \cup A_i$ are independent. \square

PROPOSITION 6. *Let A and B be bases of a matroid M . Then for every partition $B_1 \sqcup \dots \sqcup B_k = B$ there exists a partition $A_1 \sqcup \dots \sqcup A_k = A$, such that $(A \setminus A_i) \cup B_i$ are all bases for $1 \leq i \leq k$.*

PROOF. We assume that bases A, B are disjoint, and restrict to their union. Consider matroid M' equal to M , where elements of A have $k - 1$ parallel copies. So the ground set of M' equals to $A^1 \sqcup \dots \sqcup A^{k-1} \sqcup B$. Let ℓ be a lists size function such that $\ell|_{A^i} \equiv k$, and $\ell|_B \equiv 1$. It is easy to check that condition (1) of Theorem 2 is satisfied.

Let L be a list assignment of size ℓ which assigns list $\{i\}$ to elements of B_i , and list $\{1, \dots, k\}$ to elements of A and all its copies. By Theorem 2 there exists a proper coloring from these lists. Denote by C_i elements colored with i . Now let A_i contain all $a \in A$ such that no copy of a is colored with i . Then sets $(A \setminus A_i) \cup B_i$ are independent, so it is a good partition. \square

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